Non-free sections of Fano fibrations

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Height functions

$\mathbb{P}^n/\mathbb{Q}$: the n-dimensional projective space over $\mathbb{Q}$.

A height function $\mathrm{H} : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ is defined by

$$\mathrm{H}(x_0 : \cdots : x_n) = \max\{|x_0|, \cdots, |x_n|\},$$

where $x_i \in \mathbb{Z}$ and $\gcd_i(x_i) = 1$.

Note that

$$\{P \in \mathbb{P}^n(\mathbb{Q}) \mid \mathrm{H}(P) \leq T\}$$

is a finite set.

Height functions

Suppose that we are given a number field F, a projective variety X defined over F, and a Cartier divisor L on X.

Then one can assign a height function $\mathrm{H}_L : X(F) \to \mathbb{R}_{>0}$ associated to a triple (F, X, L).

When L is ample, the counting function of rational points of bounded height is well-defined:

$$\mathrm{N}_{\mathrm{H}_L}(Q, T) = \#\{P \in Q \mid \mathrm{H}_L(P) \leq T\}$$

where $Q \subset X(F)$ is any subset.

Arithmetic geometers seek the asymptotic formula of this counting function as $T \to \infty$ for an appropriate choice of Q.

Birational invariants in Manin's conjecture

F: a field of characteristic 0
X: a smooth projective variety defined over F
L: a big $\mathbb{Q}$-divisor
K_X: the canonical divisor of X

Definition

The Fujita invariant or the a-invariant is

$$a(X, L) = \min\{t \in \mathbb{R} \mid tL + K_X \in \overline{\mathrm{Eff}}^1(X)\}.$$

When L is not big, we define $a(X, L) = +\infty$.

By [Boucksom-Demailly-Păun-Peternell, '13], when L is big,

$$a(X, L) > 0 \iff K_X \text{ is not pseudo-effective}$$
$$\iff X \text{ is geometrically uniruled}$$

Birational invariants in Manin's conjecture

Assume $0 < a(X, L) < \infty$.

Definition

We define the face

$$\mathcal{F}(X, L) = \mathrm{Nef}_1(X) \cap \{(a(X, L)L + K_X).\alpha = 0\}.$$

This is an extremal face of $\mathrm{Nef}_1(X)$. We define

$$b(F, X, L) := \dim\langle \mathcal{F}(X, L)\rangle.$$

When X is singular, we take a smooth resolution $\beta : \widetilde{X} \to X$ and define

$$a(X, L) = a(\widetilde{X}, \beta^* L), \quad b(F, X, L) = b(F, \widetilde{X}, \beta^* L).$$

These are well-defined due to birational invariance of the a, b-invariants.

Birational invariants in Manin's conjecture

Example

X: a smooth Fano variety defined over F,
$L = -K_X$
Then

$$a(X, L) = 1, \quad b(F, X, L) = \dim N^1(X) = \rho(X).$$

Thin sets

F: a number field
X: a variety defined over F

A type I thin set is $V(F) \subset X(F)$ where $V \subset X$ is proper closed.

A type II thin set is $f(Y(F))$ where $f : Y \to X$ is a dominant generically finite morphism of degree ≥ 2 with Y a variety.

Example

$$\{x^2 \in \mathbb{A}^1(F) \,|\, x \in \mathbb{A}^1(F)\}$$

A thin set is any subset of the finite union of type I and II thin sets.

Manin's conjecture

Conjecture ('90–'03, Batyrev–Manin–Peyre–Tschinkel)

F: a number field
X: a smooth projective geom. rationally conn. variety over F.
$\mathcal{L}$: a big and nef $\mathbb{Q}$-divisor on X with an adelic metrizaiton
Suppose $X(F)$ is not thin. Then $\exists Z \subset X(F)$ a thin set s. t.

$$N(X(F) \setminus Z, \mathrm{H}_{\mathcal{L}}, T) \sim c(F, Z, \mathcal{L}) T^{a(X,L)} (\log T)^{b(F,X,L)-1}$$

as $T \to \infty$. Here $c(F, Z, \mathcal{L})$ is Peyre's constant introduced by Peyre for the anticanonical divisor and by Batyrev–Tschinkel for arbitrary big divisors.

The set Z is called as the exceptional set which is the main theme of this talk.

The main theorem

Theorem ('22 Lehmann–Sengupta–T)

F: a number field
X: a smooth geometrically uniruled projective variety over F
L: a big and nef divisor on X
Let $f : Y \to X$ run over all generically finite morphism over F from a smooth projective variety Y to the image such that

$$(a(X, L), b(F, X, L)) < (a(Y, f^*L,), b(F, Y, f^*L))$$

in the lexicographic order. Then the union

$$\bigcup_{f:Y \to X} f(Y(F))$$

is a thin set.

Ideas of proofs

The main ingredients of the proof are

- the minimal model program
- BAB conjecture, proved by Birkar
  i.e., the boundedness of singular Fano varieties
- Hilbert's irreducibility theorem
- systematic usage of étale fundamental groups and homotopy lifting property with a rational base point (Lemma 8.3)

Manin's conjecture over function fields

One can also consider Manin's conjecture over function fields.

We work over $\mathbb{C}$
B: a complex smooth projective curve

Definition

A Fano fibration is a morphism $\pi : \mathcal{X} \to B$ such that:

- $\mathcal{X}$ is a smooth projective complex variety
- B is a smooth projective complex curve
- π is flat with connected fibers, and
- the generic fiber of π is a Fano variety.

Moduli space of sections

$\pi : \mathcal{X} \to B$: a Fano fibration
We denote the space of sections of π by $\mathrm{Sec}(\mathcal{X}/B)$.
This consists of countably many irreducible components.

When $\pi : \mathcal{X} \to B$ is a trivial family, i.e., $\mathcal{X} = X \times B$, $\mathrm{Sec}(\mathcal{X}/B)$ can be identified with the space of morphisms $\mathrm{Mor}(B, X)$ parametrizing morphisms $s : B \to X$.

Deformation theory of sections

M : a component of $\mathrm{Sec}(\mathcal{X}/B)$

$$\dim M \geq -K_{\mathcal{X}/B}.C + (\dim \mathcal{X} - 1)(1 - g(B))$$

where $-K_{\mathcal{X}/B}$ is the relative anticanonical class, $C \in M$, and $g(B)$ is the genus of B.
The right term is called as the expected dimension.

We also have the following inequality:

$$\dim M \leq -K_{\mathcal{X}/B}.C + (\dim \mathcal{X} - 1)(1 - g(B)) + h^1(C, T_{\mathcal{X}/B}|_C).$$

In particular when $H^1(C, T_{\mathcal{X}/B}|_C) = 0$, we have the equality.

Geometric Manin's conjecture

In his hand written notes in 1988, Batyrev developed a heuristics for Manin's conjecture over global function fields.
This is relied on the following predictions about the geometry of moduli space of sections.
Geometric Manin's conjecture:

- (Exceptional set) "Pathological" families of sections are controlled by the Fujita invariant.
- (Uniqueness) There is a unique non-pathological component of $\mathrm{Sec}(\mathcal{X}/B, \alpha)$, which should be counted in Manin's conjecture, for sufficiently positive algebraic class α of sections. We call this unique component as the Manin component.
- (Stability) Manin components of sections exhibit homological or motivic stability as the degree increases.

An idea of using homological stability in Batyrev's heuristics is due to Ellenberg and Venkatesh.

Pathological components: Non-free curves

Definition

Let $\pi : \mathcal{X} \to B$ be a Fano fibration.
A section C of π is relatively free if $T_{\mathcal{X}/B}|_C$ is globally generated and $H^1(C, T_{\mathcal{X}/B}|_C) = 0$.

Main theorem I ('23, Lehamnn-Riedl-T)

$\pi : \mathcal{X} \to B$: a Fano fibration.

Then $\exists \xi = \xi(\pi)$ and $\exists T = T(\pi)$ with the following properties.
M: an irreducible component of $\mathrm{Sec}(\mathcal{X}/B)$ parametrizing a family of non-relatively-free sections C which satisfy $-K_{\mathcal{X}/B} \cdot C \geq \xi$.

$\mathcal{U}^\nu$: the normalization of the universal family over M and let $ev : \mathcal{U}^\nu \to \mathcal{X}$ denote the evaluation map.

Then either:
(I) ev is not dominant.
Then the subvariety $\mathcal{Y}$ swept out by the sections parametrized by M satisfies

$$a(\mathcal{Y}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}).$$

Main theorem I ('23, Lehamnn-Riedl-T)

(II) *ev* is dominant.
$f : \mathcal{Y} \to \mathcal{X}$: the finite part of the Stein factorization of *ev*,
Then we have

$$a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}).$$

Furthermore, there is a rational map $\phi : \mathcal{Y} \dashrightarrow \mathcal{Z}$ with the following properties: C: a general section on $\mathcal{Y}$
$\mathcal{W}$: a resolution of the main component of the closure of $\phi^{-1}(\phi(C))$.

1. We have $a(\mathcal{W}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B})$.
2. The Iitaka dimension of $K_{\mathcal{W}_\eta} - a(\mathcal{W}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{W}_\eta})f^*K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}$ is 0.
3. the strict transform of a general deformation of C in $\mathcal{W}$ is relatively free in $\mathcal{W}$.
4. The sublocus of M parametrizing deformations of C in $\mathcal{W}$ has codimension at most T in M.

Main theorem II ('23, Lehamnn-Riedl-T)

$\pi : \mathcal{X} \to B$: a Fano fibration.

1. There is a proper closed subset $\mathcal{V} \subsetneq \mathcal{X}$ such that if $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is an irreducible component parametrizing a non-dominant family of sections then the sections parametrized by M are contained in $\mathcal{V}$.
2. There is a bounded family of smooth projective B-varieties $\mathcal{Y}$ equipped with B-morphisms $f : \mathcal{Y} \to \mathcal{X}$ satisfying:
   1. f is generically finite onto its image but not birational;
   2. $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta})$;
   3. if equality of Fujita invariants is achieved, then the Iitaka dimension of $K_{\mathcal{Y}_\eta} - a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta})f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}$ is zero;

   If $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is a component that generically parametrizes non-relatively free sections of sufficiently large degree, then a general section C parametrized by M satisfies either (i) $C \subset \mathcal{V}$ or (ii) $C = f(C')$ where $f : \mathcal{Y} \to \mathcal{X}$ is in our family, and C' is a relatively free section in $\mathcal{Y}$.

Proofs of Main Theorem I (II)

We first outline the proof of case (II) of Main Theorem I.
We assume that *ev* has connected fibers so that $\mathcal{Y} = \mathcal{X}$.

We must construct a rational map ϕ from a dominant family of non-free sections.

Our strategy relies on the theory of foliations and slope stability.

The following version of the Grauert-Mulich theorem allows us to compare semistabliity of a sheaf $\mathcal{E}$ on $\mathcal{X}$ and semistability of $\mathcal{E}|_C$.

Proofs of Main Theorem I (II)

Grauer-Mulich Theorem ('23, Lehmann-Riedl-T)
X: a smooth projective variety
$\mathcal{E}$: a torsion-free sheaf on X.
M: an irreducible component of $\mathrm{Mor}(B, X)$,
$\mathcal{U}$: the universal family over M,
$\mathcal{U}^\nu$: the normalization of $\mathcal{U}$.

Suppose that the evaluation map $ev : \mathcal{U}^\nu \to X$ is dominant with connected fibers and that for some open subset $M_{red}^\circ \subset M_{red}$ the restriction of ev to the preimage of M_{red}° is flat.

Proofs of Main Theorem I (II)

For a general curve C parametrized by M, write

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{E}|_C$$

for the Harder-Narasimhan filtration of $\mathcal{E}|_C$.

Suppose that $\mathcal{E}$ is $[C]$-semistable. Then for every index i we have

$$|\mu(\mathcal{E}|_C) - \mu(\mathcal{F}_i/\mathcal{F}_{i-1})| \quad \leq (g(B)(\dim(X) - 1) + 1)^2 \mathrm{rk}(\mathcal{E}).$$

Proofs of Main Theorem I (II)

Suppose that C is a general member of a dominant family of non-relatively free sections of π whose evaluation map has connected fibers.

In this situation $T_{\mathcal{X}/B}|_C$ is generically globally generated, so the non-relatively free implies that $T_{\mathcal{X}/B}|_C$ has a low slope quotient.

The Grauert-Mulich theorem (applied to a birational model flattening the family) yields a foliation $\mathcal{F} \subset T_{\mathcal{X}}$ of large slope.

We then appeal to Campana-Păun to see that the foliation is induced by a rational map ϕ.

Proofs of Main Theorem I (II)

Theorem ('19 Campana–Păun)

Let X be a complex smooth projective variety and $\mathcal{F} \subset T_X$ be a foliation such that $\mu_\alpha^{\min}(\mathcal{F}) > 0$ for some movable class α.

Then $\mathcal{F}$ is induced by a rational map $\phi : X \dashrightarrow Z$ with rationally connected fibers.

Now we obtained a rational map $\phi : \mathcal{X} \dashrightarrow \mathcal{Z}$.

How do we compare the a-invariants?

Proofs of Main Theorem I (II)

Theorem ('23 Lemann-Riedl-T)

$\pi : \mathcal{X} \to B$: a Fano fibration.
T: a positive integer
There is some constant $\xi = \xi(\pi, T)$ with the following property.

$\psi : \mathcal{Y} \to B$: a morphism with connected fibers from a smooth projective variety $\mathcal{Y}$ and $f : \mathcal{Y} \to \mathcal{X}$ is a B-morphism that is generically finite onto its image.
N: an irreducible component of $\mathrm{Sec}(\mathcal{Y}/B)$
M: the irreducible component of $\mathrm{Sec}(\mathcal{X}/B)$ containing f_*N.

Assume that the sections C parametrized by N satisfy $-f^*K_{\mathcal{X}/B} \cdot C \geq \xi$ and that the codimension of the closure of f_*N in M is at most T. Then

$$a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}).$$

Proofs

We next outline the proof of Main Theorem II.

Main Theorem I shows that families of non-free curves come from maps $f : \mathcal{Y} \to \mathcal{X}$ such that $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{X}_\eta})$.

[Lehmann-Sengupta-T, Lemma 8.3] shows that if we base change to the algebraic closure $\overline{K(B)}$ then there are finitely many families of such maps $f_{\overline{\eta}} : \mathcal{Y}_{\overline{\eta}} \to \mathcal{X}_{\overline{\eta}}$ whose twists account for the rational point contributions of all maps of this type.

In this way the proof of Main Theorem II is reduced to the study of twists over the ground field $K(B)$.

Proofs of Main Theorem II

We systematically study the twists of a fixed map $f_\eta : \mathcal{Y}_\eta \to \mathcal{X}_\eta$.
We construct a space of twists and prove a local-to-global principle.

Lemma ('23, Lehmann-Riedl-T)

d, b: positive integers
$f_\eta : \mathcal{Y}_\eta \to \mathcal{X}_\eta$: a generically finite dominant morphism where $\mathcal{X}_\eta$ and $\mathcal{Y}_\eta$ are normal projective varieties.

S: the set of twists f_η^σ such that $\mathcal{Y}_\eta^\sigma$ and $\mathcal{Y}_\eta$ become isomorphic after a base change by a Galois extension $K(C)/K(B)$ whose degree is $\leq d$ and whose branch locus consists of at most b points.

Then there is a finite type scheme R over $\mathbb{C}$ and morphisms $\psi : \mathcal{U}_R \to R$, $g : \mathcal{U}_R \to \mathcal{X}_\eta$ such that every element $\mathcal{Y}_\eta^\sigma \in S$ is isomorphic to the fiber of ψ over some closed point $t \in R$ and $f = g|_{\psi^{-1}t}$.

Proofs of Main Theorem II

Theorem (Boundedness of twists, '23, Lehmann-Riedl-T)
$\pi : \mathcal{X} \to B$: a Fano fibration.
$\mathcal{Y}$: a normal projective variety equipped with a dominant morphism with connected fibers $\psi : \mathcal{Y} \to B$
$f : \mathcal{Y} \to \mathcal{X}$: a dominant generically finite B-morphism

$\mathcal{Y}^\sigma$: a smooth projective B-variety
$f^\sigma : \mathcal{Y}^\sigma \to \mathcal{X}$: a dominant generically finite morphism such that $f^\sigma_\eta : \mathcal{Y}^\sigma_\eta \to \mathcal{X}_\eta$ is birational to a twist of f_η.

T: a positive integer

Proofs of Main Theorem II

Then there exist constants $d = d(\mathcal{Y}/\mathcal{X})$ and $n = n(\mathcal{Y}/\mathcal{X}, T)$ with the following properties:

Suppose $\exists$ an irreducible component $N^\sigma \subset \mathrm{Sec}(\widetilde{\mathcal{Y}}^\sigma/B)$ parametrizing a dominant family of sections on $\widetilde{\mathcal{Y}}^\sigma$ such that the pushforward of N^σ has codimension at most T in a component of $\mathrm{Sec}(\mathcal{X}/B)$.

Then there exists a finite Galois morphism $B' \to B$ of degree at most d with at most n branch points such that the base changes of f_η and f_η^σ to $K(B')$ are birationally equivalent.

Proofs of Main Theorem II

How to prove such a statement?

The degree is essentially bounded by $\#\mathrm{Bir}(\mathcal{Y}_{\overline{\eta}}/\mathcal{X}_{\overline{\eta}})$

The discriminant is bounded by the number of non-vanishing local invariants.

The number of non-vanishing local invariants is bounded by the number of non-reduced fibers of $\mathcal{Y}^\sigma/B$

Such a bound can be obtained using the canonical bundle formula combined with a bound of intersection of sections against the ramification divisor.

Such a bound of intersection number follows from dimension estimate.

Examples

$\pi : \mathcal{X} \to B$: a Fano fibration over B such that $\mathcal{X}_\eta$ is a smooth cubic hypersurface of dimension ≥ 5.

Adjunction theory tells us that there is no generically finite morphism $f : \mathcal{Y}_\eta \to \mathcal{X}_\eta$ such that $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta})$.

Our main theorem I implies that general sections of sufficiently large anticanonical height are relatively free.

On weak Manin's conjecture

F: a number field
B: a smooth projective curve over F.
S: a finite set of places of F including all archimedean places
$\mathfrak{o}_{F,S}$: the ring of S-integers in F.

$\pi : X \to B$: a Fano fibration defined over F
$\widetilde{\pi} : \mathcal{X} \to \mathcal{B}$: an integral model of π over $\mathfrak{o}_{F,S}$ such that $\mathcal{X}$ and $\mathcal{B}$ are smooth over $\mathfrak{o}_{F,S}$

$V \subset X$: the Zariski closure of the union of the loci swept out by non-dominant families of sections in $\mathrm{Sec}(X/B)$.
By Main Theorem II (1) this a proper closed.
$\mathcal{V} \subset \mathcal{X}$: the flat closure of V

On weak Manin's conjecture

v: a non-archimedean place of F not contained in S
$\pi_v : X_v \to B_v$: the reduction at v which is over a finite field k_v
V_v: the reduction of V

$\mathrm{Sec}(X_v/B_v, V_v)_{\leq d}$: the open subset of $\mathrm{Sec}(X_v/B_v)$ parametrizing sections $C \not\subset V_v$ of anticanonical height $\leq d$.

Then we consider the following counting function:

$$N(X_v \setminus V_v, -K_{X_v/B_v}, d) = \#\mathrm{Sec}(X_v/B_v, V_v)_{\leq d}(k_v).$$

Weak Manin's Conjecture over $K(B_v)$ predicts that
for any $\epsilon > 0$ we have

$$N(X_v \setminus V_v, -K_{X_v/B_v}, d) = o(q_v^{d(1+\epsilon)}),$$

as $d \to \infty$ where $q_v = \#k_v$.

On weak Manin's conjecture

Theorem (Lehmann-Riedl-T, '23)

Let $F, S, \widetilde{\pi} : \mathcal{X} \to \mathcal{B}$ be as above.
Then assuming $d\epsilon > \dim X_\eta$, we have

$$\frac{N(X_v \setminus V_v, -K_{X_v/B_v}, d)}{q_v^{d(1+\epsilon)}} \to 0$$

as $v \to \infty$.

Thank you!!